Fitting nonlinear low-order models for combustion instability control

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Abstract

Tools for fitting low-complexity nonlinear models based on experimental data are examined through the example problem of finding a reduced-order model suitable for control of a combustion instability operating in a limit cycle. This proceeds in four parts; physical modeling, linear system identification, nonlinear analysis, and validation test design. It is shown how the nonlinear tools of describing functions, bifurcation methods and manifold analysis assist in developing a simple nonlinear model capable of describing the data and consistent with physical understanding. The system being modeled is a lean gas turbine combustor which exhibits a sustained mid-range (100–1000 Hz) limit cycle instability. The closed-loop experimental data does not contain a sufficiently rich spectrum for confident modeling in the first linear system identification phase. Despite the paucity of information available, a grey-box nonlinear model is created and parametrized which provides an explanation both of the limit-cycle fundamental oscillation and of a high frequency nonharmonic signal also present. The model structure is explored and various operating conditions simulated to understand the model better.

The validation and/or refinement of this model is then considered. The model validation problem is important because of the poor information content of the periodic limit cycle data. The challenge is to provide a practically feasible, small excitation to the loop to improve identifiability and to provide qualitative tests of model performance. We examine this problem by considering the nonlinear dynamics of the model class and feasible excitation mechanisms.

Keywords: Nonlinear modeling; Combustion instability; System identification; Bifurcation analysis

1. Introduction

Tools for the reduced order nonlinear modeling of an unstable combustion process from gas turbines are developed. The combustion instabilities which we consider are medium-range (100–1000 Hz) oscillations attributable to the nonlinear interaction between chamber acoustics and the heat release rate, most likely excited by turbulence or vortex production. These instability mechanisms appear when the combustor is operated at low fuel-to-air ratios (or low equivalence ratios,\textsuperscript{2} $\phi$). These operating points are commercially and environmentally advantageous because they lead to complete combustion at low temperatures, which in turn leads to the production of less NO\textsubscript{X} compounds and to lower demands on equipment properties. Amelioration of these instabilities by feedback control is therefore of direct commercial interest. The problem of modeling and understanding this phenomenon in lean premixed combustors has received a great deal of attention in recent years, see, for example (Annaswamy).\textsuperscript{3}

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\textsuperscript{2}Equivalence ratio ($\phi$) is defined as, $\phi = (W_f/W_a)/(W_f/W_a)\times 100$, where $W_f$ and $W_a$ are the fuel and air mass flows injected at the nozzle per unit of time. The subscript $s$ denotes the “stoichiometric value” at which complete consumption of both fuel and oxygen would occur. Stoichiometric combustion is characterized by an equivalence ratio $\phi = 1$.\n
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The task of modeling this instability is inherently difficult. The combustion chamber signals are persistently periodic and there is no measurable external excitation signal. This suggests that the instability might be generated by a nonlinear closed loop perturbed only by stochastic disturbances. Further, these strongly periodic signals which occur during normal operating conditions do not contain a sufficiently rich spectrum for reliable model estimation using standard tools.

The model we seek to fit to this process needs to capture the dynamical system linking the actuator, be it a fuel flow valve, acoustic actuator such as a loudspeaker, or other airflow manipulator, and the sensor — most likely a single pressure sensor on the wall of the chamber. This model should facilitate the subsequent design of a feedback controller. With this architecture and the spatial and frequency response limitations of actuation and sensing, there will be no capability of such a model to capture the full four-dimensional fluid mechanics of the combustion process. Unsteady fluid dynamics (e.g. vortex shedding, flame area variations, and different fuel mixing zones) cannot be observed nor controlled with this architecture and are left outside the model. Thus the model is deliberately low order or reduced complexity compared with a full partial differential equation (PDE) approach. Indeed the capacity of such low-order models to capture the dominant dynamics has been supported by several combustion experiments (Anderson, Sowa, & Morford, 1998). The underlying question is whether such a model is adequate for the design of a stabilizing controller. If not, then it is unlikely that this control architecture would suffice to control these instabilities.

The purpose of this model is ultimately to be used to generate a feedback controller capable of stabilizing (or at least ameliorating) the instability, which manifests itself early on as a limit cycle in the flame position and other derived physical combustion chamber measurements. The model is initially fitted using system identification techniques and several sets of quasi-periodic data measured from the limit-cycling system at different operating conditions (Savaresi, Bitmead, & Dunstan, 2001). The tools used in this earlier work are predominantly from the linear system regime and fall into the categories of: frequency estimation, delay estimation, parametric linear system identification, together with the estimation of a memoryless nonlinear function. In this paper, we extend this through the inclusion of truly nonlinear tools such as; describing function methods, bifurcations, manifold analysis and nonlinear oscillations.

Our aim is to explore these nonlinear methods as a means to assist in model identification and validation. Specifically, in the linearized analysis of (Savaresi et al., 2001), questions of model validity arise because of the poor information content of data derived from the periodic signals from a limit-cycling system. It is not apparent how the multiple experiments may be used to improve the confidence in model fit. Nor do the tools exist to understand fully the expected effects of altering the experimental conditions through, say, the injection of an external excitation signal.

The tools we explore here are those of linear system identification (Section 3), nonlinear system identification (Section 4) and validation experiment design (Section 5). In this context of fitting nonlinear combustion instabilities, we concentrate on the development of sound methodologies rather than modeling a specific engine’s combustor. The underlying instability mechanism of interacting acoustics and heat-release would appear to be generic (Hill & Peterson, 1992) but the appropriate tools of system identification are yet rudimentary.

2. Physics and data

The application producing the intermediate frequency instability in this case is a lean premixed gas turbine combustor which transitions to a limit cycling behavior at low equivalence ratios. We consider here a generic low-order physical model structure proposed in Peracchio and Proscia (1998), which we then relate to experimental data acquired from the UTRC/DARPA single nozzle rig—an isolated section of a full gas turbine engine. The data recorded were the sensed downstream pressure, $p_k$, in the combustor and the heat release rate, $q_k$, measured by an optical sensor. The experimental layout is illustrated in Fig. 1. Data was collected from six autonomous experiments conducted at different equivalence ratios within the region where the system is limit cycling ($\phi_1 = 0.56, \phi_2 = 0.53, \phi_3 = 0.51, \phi_4 = 0.49, \phi_5 = 0.47, \phi_6 = 0.45$). By autonomous we mean free from external excitation and exhibiting sustained self-generated periodic oscillations. For each experiment the pressure, $p_1$, and heat release rate, $q_1$, were sampled at 5000 Hz and 16 000 samples were logged after the transient phase of the limit cycle had diminished. The measured pressure and heat release rate are denoted $p_k$ and $q_k$ with $k$ to indicate the sampled data time index and the presence of sensor and sampling filters, drawing the distinction from the actual pressure, $\bar{p}_1$, and heat release rate, $\bar{q}_1$.

The six sets of data are analyzed under the assumption that they are cyclo-stationary signals. This is well satisfied by the signals of experiments $\phi_1$–$\phi_4$, but not fully true for experiments $\phi_5$–$\phi_6$.

and $\phi_k$ which show some time-dependent features, possibly the beginning transition to severe chaotic instability.

Fig. 2 shows an 80 ms time segment of the measured pressure ($\bar{p}_k$) and heat release rate data ($\bar{q}_k$) from Experiment $\phi_1$. Fig. 3 shows the pressure and heat release rate spectra from Experiment $\phi_1$. From a qualitative analysis of the $\bar{p}_k$ and $\bar{q}_k$ data over the six experiments, the following preliminary conclusions can be drawn.

- $\bar{p}_k$ and $\bar{q}_k$ are characterized by a nondamped oscillation; since we assume that there is no external excitation, the system must be affected by a limit-cycle phenomenon. This is possible only if the system is characterized by nonlinear dynamics.
- The data is almost periodic. It is dominated by sinusoids at 210 and 740 Hz, plus harmonics of 210 Hz and some small broad-band noise component. Note that it is almost periodic (and not periodic) because the 210 and 740 Hz signals are not harmonically related.
- The fundamental (210 Hz for $\phi_1$) oscillation increases in frequency for increasing values of $\phi$ while the equivalent of the 740 Hz oscillation appears not to shift.
- The $\bar{q}_k$ signal is deficient of high frequency components, which we attribute to the heat release rate sensor exhibiting a low pass filtering effect. This has been confirmed by subsequent enquiry.

The low-order model of Peracchio and Proscia (1998) was developed from the physical principles believed to be responsible for producing the combustion instability. The model relates the interaction of the bulk acoustic mode ($p_t$) in the combustor to the heat release rate ($q_t$) of the flame front as a function of equivalence ratio ($\phi$), for a range of low equivalence ratios ($0.45 > \phi > \ldots$).
The model, shown in Fig. 4, has the following structure:

\[
\dot{p}_t + 2\omega\xi p_t + \omega^2 p_t = N \frac{d}{dt} \Phi[p(t - \tau)],
\]

(1)

where \(p_t\) is the pressure at the burning plane, \(\Phi[\cdot]\) is a 1-input 1-output static nonlinear mapping, which physical considerations suggest should be characterized by a nonpositive slope, \(q_t\) is the heat release rate at the burning plane, \(\omega, \xi,\) and \(N\) are the central frequency, damping and scaling factor of the second-order oscillator representing the fundamental acoustic mode of the combustor, \(\tau\) is a time delay—specifically, it is the time taken by the mixed gas at the nozzle to reach the burning plane at the end of the combustion chamber. LPF is a low pass filter included into the loop to ensure well-posedness. This is not part of the original Peracchio and Proscia formulation but such a mechanism for high frequency attenuation needs to be assumed somewhere in the process. The bandwidth of this filter is presumed to be above the 2500 Hz cutoff of the sampled data.

Eq. (1) is derived by the ordinary differential equations which couple the bulk density, pressure and velocity in the combustion chamber, and by an energy-balance equation. Notice that (1) does not contain partial derivatives. It is obtained by expanding the solution of the general model, which is a distributed-parameter model in terms of the orthogonal acoustic modes of the systems, defined by an eigenvalue problem, and picking up the first mode only. This model structure provides the basis for a significant data analysis by Murray et al. (1998) as extended by the authors (Savaresi et al., 2001), which we further develop in subsequent sections.

The candidate model structure that we use as the basis for our fitting of the data is illustrated in Fig. 5, it incorporates the model of Peracchio and Proscia (1998) extended by the following components.

A third-harmonic oscillator in parallel with the original oscillator.

An anti-aliasing filter (AAF) has been incorporated where each signal is sampled. This corresponds to the Nyquist sampling frequency of the data acquisition hardware, 2500 Hz.
A sensor low-pass filter (LPF) is included as part of the $q_t$ measurement system. The 740 Hz component of $p_t$ demands the presence of a similar component in $q_t$, which is not observed in the $\tilde{q}_k$ signal. This disparity is accommodated by the low-pass filter in the heat release rate sensor path, which is supported by the sensor physics.

We note that the system depicted in Fig. 5 is of a feasible structure to produce sustained nonlinear oscillations according to, say, describing function analysis. We shall use precisely these methods as part of our identification approach. The prima facie evidence is that this model structure has the capability to demonstrate the observed signal behavior while conforming to physical principles.

The system identification task which we consider is to use the observed data to fit the unknown parameters of the system structure in Fig. 5. Our approach is within the framework of Grey-Box Modeling, a term introduced by (Bohlin, 1995; Bohlin & Graebe, 1995) to characterize a model derived from a combination of a priori process knowledge and experimental data. In the next section, we look at the linear system identification tools available and the results they produce, which in turn serve as a starting point for the nonlinear analysis.

3. Linear system identification tools

The model of Fig. 5 is nonlinear and is parametrized by five constants, $\tau, M, N, \xi$ and $\omega$, in the linear part and an unknown memoryless function $\Phi[\cdot]$. As illustrated in Fig. 3, the spectral content of $\tilde{p}_k$ is supported well by two frequencies, 210 and 740 Hz, with two other harmonic frequencies, 420 and 630 Hz, barely detectable. The spectral content of $\tilde{q}_k$ is even more deficient because of the low-pass filtering. The prospects for identifying accurately all five parameters from such uninformative data is problematic. However, we shall see that linear systems identification methods can decouple some parts of the estimation—notably that of $\tau$ and $\Phi[\cdot]$. Questions of validation and confidence for the linear part are left unanswered by these methods. Nevertheless they form the core of the work of (Murray et al., 1998; Savaresi et al., 2001), which we shall extend shortly using nonlinear tools.

The measured data are limited to the pair of almost periodic signals $\tilde{p}_k$ and $\tilde{q}_k$. The linear identification approach derives from Fig. 5 as follows.

- Consider the fundamental, 210 Hz, component of $p_t$ and the related fundamental of $q_t$. The phase relationship between these two should yield $\tau$. This property is preserved in the $\tilde{p}_k$ and $\tilde{q}_k$ data.

- Once $\tau$ is estimated, the signal at the input to $\Phi[\cdot]$ can be reconstructed from $\tilde{p}_k$. This allows the memoryless function to be plotted.

- The relationship between $\tilde{q}_k$ and $\tilde{p}_k$ is a linear system, which might be estimated using standard identification methods.

We now briefly present the details.

3.1. Frequency and time-delay estimation

In order to access the fundamental components of $p_t$ and $q_t$, we need first to estimate the frequency of the $\sim 210$ Hz signal. This is performed using a periodogram maximizer on each set of experimental data. This is achieved in Savaresi et al. (2001) using FFT and Kalman filtering methods. As a by-product of both approaches the phase shift between $\tilde{p}_k$ and $\tilde{q}_k$ fundamentals is estimated.

The time delay estimate, $\tau$, is computed by dividing the difference in phase between the fundamental harmonics of the $\tilde{p}_k$ and $\tilde{q}_k$ signals by $\omega$. Beginning with the $p_t$ fundamental and working clockwise around the loop, the differentiator and nonlinearity add $3\pi/2$ to the phase of $p_t$, with the remainder of the phase
difference being due to the delay. We assume that the LPF affecting \( q_t \) rolls off outside the bandwidth of the fundamental.

This technique results in \( \tau = 3.47 \) ms. Interestingly, this value is constant for all \( \phi \), which strongly supports the proposed model structure incorporating a fixed time delay.

### 3.2. Nonlinearity estimation

The memoryless nonlinearity, \( \Phi[] \), is charted by reconstructing the input signal to the nonlinearity by; splitting \( \tilde{p}_k \) into its periodic components, mapping the amplitude and phase of each component as affected by the delay and differentiator, and recombining to produce the input to \( \Phi[] \). This signal is plotted pointwise against \( \tilde{q}_k \) as a scatter plot. The “cloud” of points formed by the scatter plot suggests a negative slope saturation. A continuously differentiable function is fitted giving the negative slope saturation with a DC offset shown in Fig. 6. The slope and range of \( \Phi[] \) vary with equivalence ratio, \( \phi \), i.e. with experiment, but maintain a similar form. One can consider Fig. 6 as either a negative slope saturation in positive feedback with linear portion of the loop, or as a positive slope saturation in negative feedback. We shall treat it as the former, although the latter form is commonly seen in nonlinear methods such as describing function analysis.

An independent verification of the time-delay estimate is provided by the evident deblurring of the cloud of points of this chart as the time delay is varied.

### 3.3. Linear system estimation

It is possible to specify uniquely at most two independent linear system parameters per frequency point of signal support. Here, we have four parameters \( M, N, \xi \) and \( \omega \) and a set of six independent limit-cycling experiments, one for each value of \( \phi \). The frequency content of the data is too poor to permit confident estimation of all four of these parameters from any one experiment. More information needs to be injected or invoked.

The following reasoning is used in Savaresi et al. (2001):

- The natural frequency of the acoustics is linked to the geometrical configuration of the combustion chamber and should not vary with \( \phi \).
- The coupled oscillators aim to model a persistent acoustic mode, for which the damping ratio, \( \xi \), should be small and perhaps slowly varying with \( \phi \).
- One would expect that the damping ratio, \( \xi \), would increase with \( \phi \).

Combining these qualitative interpretations and the success of permissible loop phase values at the fundamental oscillating frequency over the range of six experiments allows a more confident estimate of the oscillator parameters for \( \phi_1 \): \( \omega = 2\pi \times 206 \text{ rad} \), \( \xi = 0.2 \), \( N = 1 \), \( M = 1 \). Note that \( \omega \) does not equal the frequency of the fundamental oscillation (\( \omega \)) in the data. But it is close.

Although, we can identify the set of five parameters of the linear part of the model, we have little confidence in their accuracy. The spectral content is insufficiently rich (or at least poorly conditioned), the variations between experiments are small, and we are trying to use this data to fit five parameters. We are concerned about the quality of the fit, particularly for \( M, N, \xi \) and \( \omega \). When identification is conducted over different data sets for varying \( \phi \), it becomes difficult to distinguish changes in \( \Phi[] \) from changes in the parameters. We next ask how truly nonlinear methods might be applied to develop improved model confidence.

Before introducing the nonlinear analysis, we can make some remarks about the Nyquist diagram of the identified linear plant which is in closed loop with the memoryless nonlinearity. Fig. 7 shows the Nyquist diagram for the complete linear forward path from the output of the nonlinearity back to its input. Frequency varies from DC through to 2500 Hz, the Nyquist frequency. Observe that the outer most negative real axis crossings (marked with black dots) occur at values of 210 and 740 Hz, the components present in the data. We note that this fact was not used in deriving the linear part of the model. That a model built for harmonic balancing at the fundamental frequency of the limit cycle is able to foreshadow the possible existence of the nonharmonic 740 Hz component is very surprising. Also note that the effect of the delay is to cause unlimited winding of the curve as frequency increases.
4. Nonlinear system identification methods

Part of the validation process for an identified model is to explore whether it possesses the capacity to describe the data with some degree of fidelity. Subsequent analyses should then be based on testing the predictive power of the model in a realistic scenario. In this section, we apply a variety of nonlinear tools to probe the nature of the identified model before demonstrating its surprising ability to reproduce the data signals. The principal issue which we explore is the coexistence of the two nonharmonically related frequencies 210 and 740 Hz.

4.1. Describing function analysis

We note the candidate system structure from Fig. 5 possesses a memoryless nonlinearity in feedback with a strictly linear system. The closed-loop signal data is almost periodic. This is precisely the province of describing function methods (Gelb & Vander Velde, 1968; Khalil, 1996; Holtzman, 1970), which represent a linearized stability analysis of the system and are capable of predicting stable oscillations.

The describing function $\Psi(a, \omega)$ of a nonlinearity $\Phi[\cdot]$ is defined as the amplitude- and frequency-dependent effective complex gain of the nonlinearity when it is excited by an input $a \sin(\omega t)$ and the fundamental component alone of the output, $A \sin(\omega t + \theta)$, is considered. Thus,

$$\Psi(a, \omega) = \frac{A(a, \omega)}{a} e^{j(\omega t + \theta)}.$$

The describing function $\Phi[\cdot]$ in the case of the combustor is a memoryless function, for which the describing function is real and frequency-independent,

$$\Psi(a) = \frac{A(a)}{a},$$

where $A(a)$ contains the sign of the gain. To predict which oscillations may be sustained in the loop it is necessary to solve the first-order harmonic balance equation

$$G(j\omega)\Psi(a) + 1 = 0 \quad \text{or} \quad G(j\omega) = -\frac{1}{\Psi(a)} \quad (2)$$

where $G(j\omega)$ is the grouped linear portion of the loop shown in Fig. 5.

The Nyquist diagram of $G(j\omega)$ and the inverse of the describing function $-\Psi^{-1}(a)$ are plotted on the same complex plane. Points of intersection, corresponding to solutions of (2), provide candidates for sustained oscillation, with the frequency $\omega$ from $G(j\omega)$ and the amplitude $a$ from $\Psi$ yielding the corresponding frequency and amplitude of oscillation. Encirclement conditions yield local stability information. The method is based on linearization and relies on the dominance of
the closed-loop signals by the first mode of response. This translates into a requirement that the linear system should be low pass, which is evidently untrue here. Further, the system should be finite-dimensional, which is also false because of the presence of the delay term. Nevertheless, formally we may proceed since this leads to some interesting observations about the system. The describing function of the identified function, $\Phi[]$, is real and negative, and is given by

$$\Psi(a) = -|K_{eff}(a)|^{-1},$$

where $K_{eff}(a)$ is the effective gain of the nonlinearity for signals of amplitude $a$. The identified $\Phi[]$ of Fig. 6 has an effective gain of $-0.645$ for small signals, while the saturation diminishes the gain to zero for high amplitude inputs. Thus $-\Psi^{-1}(a)$ lies along the positive real axis from $-\infty$ to $-1.55$, corresponding to infinite amplitudes down to small amplitudes, respectively.

Fig. 8 shows detail of the negative real axis section of the Nyquist diagram of the linear part parametrized by frequency, together with $-\Psi^{-1}(a)$. The negative real axis crossings are labeled 1–9, and point A represents the operating point of the identified system (i.e. $|K_{eff}(a)| = 0.645$, $-\Psi^{-1}(a) = -1.55$). Several features are immediately apparent.

- The describing function intersects the Nyquist plot (with range 0–2500 Hz) eight times. The first two of these (points 1 and 2) correspond to the two frequencies visible in the data spectrum, viz. 210 and 740 Hz. However, the describing function method can only predict the possible occurrence of a single oscillation and not the joint appearance of two nonharmonic frequencies.
- Using the number of clockwise encirclements to gauge stability, all points of intersection with the Nyquist plot, except the first one (740 Hz), should yield unstable oscillations. The oscillation at 740 Hz should be stable.
- The describing function does not reach the Nyquist crossing point at 490 Hz (point 9).

Employing simulation tools we can test the describing function predictions against the behavior of the identified model. The simulated $\hat{p}_k$ spectrum produces the stable oscillation at 740 Hz, while the 210 Hz oscillation is absent, in agreement with the describing function stability analysis. While it is comforting to see that the describing function analysis and the model simulations agree, there still remains the issue that both frequencies (210 and 740 Hz) are simultaneously present in the data, and as yet we have not been able to provide the tools with which to predict this. However, bifurcation analysis of the identified model reveals more structure, which helps to understand this possible behavior.

4.2. Bifurcation analysis

In order to study in greater detail the dynamics predicted by describing function analysis, we introduce
into the identified model an adjustable loop gain, $\mu$, and ask how the dynamic properties of the loop change qualitatively with $\mu$. This is a bifurcation analysis (Guckenheimer & Holmes, 1983), for which the system is depicted in Fig. 9. The bifurcation diagram is shown in Fig. 10. The horizontal axis depicts the gain parameter $\mu$ and the vertical axis shows the amplitude of stationary points and orbits of the system. Each branch of the bifurcation diagram indicates the stability (solid line) or instability (dotted line) of the associated orbit at the particular parameter value. Fig. 11 exposes the inner detail of Fig. 10. A total of nine bifurcation branches are considered:

- Branch 1 is the steady state solution with amplitude zero, which is stable for small parameter values (large equivalence ratios $\phi$) and unstable for large parameter values (low $\phi$ or lean mixtures).
- Branches 2–9 are periodic orbits which bifurcate from the equilibrium point solution. They correspond in frequency and loop gain value to the Nyquist crossing points 1–8.

Table 1 summarizes the main details from Fig. 10. Referring to Figs. 10 and 11 and Table 1, we now walk through the results.

- At $\mu = 0.30$ we have a stable fixed point indicating the combustion process would not exhibit limit cycling behavior, consisting of a stationary flame front. From our knowledge of how the nonlinearity gain, $K_{\text{eff}}$, varies with $\phi$, we know this corresponds to a high equivalence ratio, and thus a stable combustion operating point.
- As we increase the gain, $\mu$, corresponding to a decrease in the equivalence ratio, $\phi$, this fixed point remains but becomes unstable above $\mu = 0.51$—depicted by the horizontal line changing from solid to dashed.
- At $\mu = 0.51$ a stable orbit of 740 Hz appears and grows in amplitude with $\mu$. This is a Hopf bifurcation (Guckenheimer & Holmes, 1983) and is illustrated by the roughly parabolic curve (solid) emanating from the steady state solution at $\mu = 0.51$. This 740 Hz oscillation grows in amplitude with $\mu$ and remains stable until $\mu = 1.18$, where it transitions into an unstable orbit, shown by the dotted line. Along this branch the frequency remains almost constant. Interestingly, the prediction by describing function analysis of a stable oscillation at 740 Hz occurring at $|K_{\text{eff}}| = 3.06 (\mu = 0.51)$ is correct.
- At the point $\mu = 0.61$ a second Hopf bifurcation occurs producing an unstable periodic orbit at 210 Hz. Above $\mu = 0.74$ this orbit becomes stable and continues to grow in amplitude with $\mu$. The occurrence and instability of this limit cycle concurs with the predictions of describing function analysis. Describing function analysis deals with local stability results, and so the ultimate stability of the limit cycle is outside the capabilities of describing function analysis to predict.
- Six more Hopf bifurcations occur at various frequencies between 1000 and 2500 Hz as outlined in Table 1. They all begin as unstable orbits, and, over the parameter range evaluated, only the orbit at 1009 Hz transitions to a stable oscillation at $\mu = 1.56$.
- The operating point for identification is shown by the thin vertical dashed line at $\mu = 1.0$. Thus, the predicted measurements should contain either the 210 Hz limit cycle or the 740 Hz limit cycle, depending upon initial conditions. These observations hold for this system over all equivalence ratios pertaining to the experiments conducted. (Although, the detailed parameters of the identified linear part do alter with $\phi$.)
Note that the behavior of the system is structurally stable across a range of equivalence ratios, as is borne out by the sequence of experiments. This is corroborated by the bifurcation diagram and confirms part of the describing function analysis. Indeed this full bifurcation picture is the sound alternative to the indicative (local) analysis of describing function methods.

The results for this section were calculated using the delay differential equation package “DDE-BIFTOOL” (Engelborghs, 2000). This very useful computational tool, together with the methods of bifurcation analysis, have allowed us to extend the describing function methods to describe possible coexisting periodic equilibria describing the dynamics.

To gain still further insight, we next consider the transient portion of the simulated response of the identified model. We do this by plotting part of the state phase portrait of the linear system.

### 4.3. Phase portrait analysis

The simulated system response for $\mu = 0.62$ is shown in the time domain in Fig. 12. The corresponding spectrogram (time–frequency plot) is in Fig. 13. At the beginning of the response both 210 and 740 Hz signals are present however, after $\approx 400$ ms the 740 Hz signal remains and the 210 Hz signal disappears.

This indicates that, for this gain, the 740 Hz limit cycle is stable but there exists an unstable orbit at 210 Hz. Let us turn to the detailed bifurcation diagram at this gain, shown in Fig. 14. The simulation operating point, $\mu = 0.62$ is shown dotted. At this parameter value, we would expect a stable 740 Hz oscillation and unstable 210 Hz orbit—exactly as the simulation produced. Thus describing function analysis and bifurcation analysis both support this observation that a single 740 Hz periodic orbit exists and is stable. For increasing values of $\mu$, the 210 Hz mode becomes more persistent until (as shown in Fig. 10) it dominates.

Fig. 15 shows the two-dimensional projection of the phase portrait for $\mu = 0.62$. (That is, it plots two of the states from the linear acoustic part of the system against each other as they evolve in time.) Experimentally, all solutions tend to the center of Fig. 15 appear to be attractive in the large. Fig. 16 shows the inner-most portion of Fig. 15. Notice that within the limit set there

<table>
<thead>
<tr>
<th>Point No.</th>
<th>Bifurcation Pt $\mu$, stability</th>
<th>Freq. (Hz)</th>
<th>Stability at $\mu = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.51, stable</td>
<td>740</td>
<td>stable</td>
</tr>
<tr>
<td>2</td>
<td>0.61, unstable</td>
<td>210</td>
<td>stable</td>
</tr>
<tr>
<td>3</td>
<td>0.64, unstable</td>
<td>1009</td>
<td>unstable</td>
</tr>
<tr>
<td>4</td>
<td>0.72, unstable</td>
<td>1290</td>
<td>unstable</td>
</tr>
<tr>
<td>5</td>
<td>0.77, unstable</td>
<td>1573</td>
<td>unstable</td>
</tr>
<tr>
<td>6</td>
<td>0.81, unstable</td>
<td>1857</td>
<td>unstable</td>
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</tr>
<tr>
<td>8</td>
<td>0.88, unstable</td>
<td>2427</td>
<td>unstable</td>
</tr>
</tbody>
</table>

Fig. 11. Zoomed inner detail of Fig. 10.
is the 740 Hz periodic orbit attractor (the dark outer ring), along with 210 Hz resonant behavior (the inner ring structure).

This result demonstrates that although one of the attractors is unstable, the system trajectory (from certain initial conditions) still incorporates transitioning to the...
unstable manifold (210 Hz), and then off onto the stable manifold (740 Hz), reinforcing the image from Fig. 12.

Now return to the system operating point, $\mu = 1.0$, where both stable manifolds are present. What appears to be important in this region is the coexistence for certain values of $\mu$ of two nearby stable orbits, since this gives a clue to how such a model can fit the data. In the next section, we pose the question, “Is it possible that

Fig. 14. Section of bifurcation diagram from Fig. 10, showing $\mu = 0.62$.

Fig. 15. Phase portrait for states $x_1$ and $x_2$ with $\mu = 0.62$. 
noise excitation might make the states transition from one stable manifold to the other continually—thereby sustaining the presence of two nonharmonic frequencies in the data?"

4.4. Stochastic perturbation model

From Fig. 3, the data exhibit the persistent presence of two stationary periodic orbits, 210 and 740 Hz, together with a small broad-band noise component. Yet each of the nonlinear analytical tools employed above—describing functions, bifurcations and manifold studies—is capable only of yielding at most a single period. Can the small noise component cause the coexistence of the two attractors?

We simulate the behavior of the identified system with the inclusion of a small noise source adding into the loop. The spectrum of this response is illustrated in Fig. 17 alongside that of the data. The concordance is remarkable, as is the ability of this modification to generate the persistence of both dominant spectral components.

To understand the behavior of this nonlinear system with two attractors excited by noise, we turn to the work of (Knobloch & Weiss, 1989). Subject to technical conditions on the attractors—"dissipativity" or phase space volume contraction onto attractors—and on the noise—"fluctuation" or controllability from the noise input—the effect of the noise can be to force continuous transitions between the attractors which permits both frequencies to coexist. The net result is that the noise causes time sharing between the attractors while introducing some broadband noise to the signals and a random walk in phase.

Should the model be validated, then the satisfaction of these dissipation and fluctuation conditions could provide a useful starting point for control design. However, at this stage we have simply provided evidence that this model could be capable of reproducing the experimental data.

These observations reinforce the acceptability of the model from the standpoint of being able to explain the data with very low complexity. We have yet to test the model's facility for control design.

5. Validation experiment design

The identification and simulation performed so far may spark some enthusiasm for the model. However, we now seek to determine experiments to test more fully the proposed structure. In particular, our aims are:

(a) to generate more information-rich signals within the loop in order that parameter identifiability be improved,
(b) to explore the appearance (or nonappearance) of qualitative dynamical phenomena as predicted by the existing model structure and parameters, with a
view to discriminating whether the model is able to continue to describe adequately the data,
(c) to derive from these tests model information which is useful for the subsequent design of feedback controllers.

We consider how one might affect the loop signals by controlled external manipulation—subject to the limitation of needing to conduct physically feasible experiments. There are several possibilities: air/fuel modulation, combustion chamber modifications and direct acoustic actuation. Of these only air/fuel modulation has sufficient authority and flexibility at this stage.

Modulating the air or fuel mass flows affects the equivalence ratio and should produce corresponding variations in the heat release rate $q_t$. However, modulation variations of this type would need to be small and limited to less than 200 Hz bandwidth, because the high frequency modulation of fuel or air flow is difficult to achieve reliably. This technique has the added advantage of exploring a likely control actuation mechanism simultaneously with the model validation.

Within the feasible region of signal bandwidth, we are restricted to only several types of signals for exciting the closed-loop system. Notably: band-limited noise, single and multiple sinusoids, and chirps. For each of these signals, we are further limited to both small signal amplitudes and to moderate frequencies which are neither too high to actuate nor too low to yield informative data.

The stochastic theory of (Knobloch & Weiss, 1989) describes how the level of noise perturbing a system affects the frequency of transitions between multiple attractors. With increased noise power, more time is shared between the attractors and the theory predicts that this would manifest itself in our model through the transfer of energy from the 210 Hz mode to the 740 Hz mode. Increasing the introduced noise power by a factor of 10 over that used to generate Fig. 17 produces the simulated system plot of Fig. 18. Here, we see that the relative magnitude of the spectral peaks is almost equal. This experiment has also widened the spectral content of
the $p_t$, which would assist the quality of the identification. However, the actuation limitations on bandwidth militate against white noise injection and force us to consider smoother signals.

5.2. Additive sinusoid and chirp injection

Fig. 19 shows the spectrogram of the simulated pressure signal when a linear chirp is introduced additively into the heat release rate, $q_t$. The level of this chirp signal is 5% of the $q_t$ signal and its range is from 50 to 200 Hz over 0.8 s. At this rate of change of frequency, the chirp effectively operates as a slowly scanning single-frequency sinusoid when compared to the number of cycles explored by the underlying attractors.

The spectrogram illustrates several features and masks some others. The interaction between the chirp and the attractors mostly occurs with the lower frequency 210 Hz attractor although the 740 Hz oscillation does increase in energy. The coupling between the chirp and the 210 Hz signal depends very much on frequency but, typically, is a destructive interference until the signals coincide in frequency. The information content of the signals is improved.

With the injection of a single frequency sinusoid, these features become more pronounced and dependent upon the specific injection frequency. Thus injection at 45 Hz almost extinguishes the 210 Hz oscillation while injection at 50 Hz has less effect. These are complex resonance phenomena. It is to be expected, however, that any feasible feedback control strategy would have to use phase-locked feedback between the pressure sensor and the fuel/air modulation, which was not done in this simulation trial.

With chirp and sinusoidal signals, one nonlinear effect was apparent: low level excitation of the system does not affect the dynamical properties of the oscillations. At $q_t$ excitation signal levels below about 4% there was no appreciable change to the simulated loop signals. Above this level, those effects indicated above became apparent.

5.3. Modeling sinusoid and chirp injection

Our aim in injecting an excitation signal to the loop via small modulation of the equivalence ratio is to test the predictive capability of the model and to improve the identifiability of the subsystems. In order to design an adequate experiment, we need to understand what to expect from the model and how this might be manifested in the real system.

Modulating equivalence ratio should have a multiplicative effect on the heat release rate, $q_t$. This is shown in Fig. 20 with the fuel/air modulation signal, $f$, entering the heat release rate multiplicatively. Signal $f$ is filtered by transfer function $G_f$ to produce signal $w$. Then $w$ is multiplied by the heat release rate to produce the driver.
signal $d$, which enters the loop additively. A stochastic component $n$ has also been added, along with gain $K_n$, as described in the stochastic perturbation section. For sinusoidal $f$, this model now has the capability of producing not only the oscillations at 210 and 740 Hz but also side-lobes and other shifted harmonics. For small signals, these additive and multiplicative models are close, but the latter model captures the side-lobes explicitly.

This increased frequency content, although providing a richer spectrum for fitting parameters, also means another set of tools will be required to understand the behavior of the new model. The describing function tools only indicate the characteristics of a single frequency, so now we need to use harmonic balancing techniques to understand fully which frequencies are sustained around the loop. The basis for future research in this area will be designing new nonlinear system identification tools suitable for this type of actuation.

The recommended feasible experiment is to use a chirp excitation to modulate the fuel flow (say 5%) from about 50 to 200 Hz. The validation test is to detect the energy transfer between modes of the system, and determine which of the above scenarios is exhibited by the system response.

6. Conclusions

We have studied in four phases the fitting of a low-complexity nonlinear model to data from a limit-cycling combustor. The first phase of physical modeling develops the closed-loop reducing-order structure parametrized by a small set of unknowns. The second phase uses the tools of linear system identification for fitting the various subsystems of the loop by estimation of the unknown parameters. The third phase, which is the centerpiece of this work, then applies the nonlinear system tools which help to complete the fit as measured by the model's being able to explain the data. The fourth phase then uses the identified nonlinear model to propose new discriminating identification tests for validation for control design.

The difficulties presented to the linear system tools in the parametric estimation part of this study are symptomatic of identification from limit cycle data;  
- the data are closed-loop and periodic and thus not particularly rich in information content,  
- there is no measured external excitation signal to provide an independent reference for identification,  
- the linear system methods cannot explicitly take into account the property that the closed-loop system limit cycles.

As a result, this phase of the identification tends to be reductionist in its approach, focusing on the subsystems without a view to the nature of the larger problem and its known behavior.

The introduction of describing functions, bifurcation methods and manifold analysis provides a means for incorporating the subsystem fitting into the study of the closed-loop dynamics. In this particular problem of the combustion instability, the principle information which we seek to preserve is the presence of both dominant sinusoidal modes in the pressure data. These tools assisted us in formulating the noise forcing term and in selecting the gain of the linear part.

The extension of model fitting to model validation is briefly treated based on combining the model with the actuation and sensing constraints. The challenge here is
to generate a discriminatory control-relevant validation test. For this purpose, we have found that a chirp modulation to the fuel flow should provide useful information. This experiment remains to be performed at this stage.

7. Unlinked reference

Casas (1999).

References


