Solution of Linear State-space Systems

Homogeneous \((u=0)\) LTV systems first

**Theorem (Peano-Baker series)**

The unique solution to \(\dot{x}(t) = A(t)x(t), \quad x(t_0) = x_0 \in \mathbb{R}^n\) is given by \(x(t) = \Phi(t, t_0)x_0\)

where \(\Phi(t, t_0) = I + \int_{t_0}^{t} A(s_1) \, ds_1 + \int_{t_0}^{t} A(s_1) \left[ \int_{t_0}^{s_1} A(s_2) \, ds_2 \right] \, ds_1 \)

\[+ \int_{t_0}^{t} A(s_1) \left[ \int_{t_0}^{s_1} A(s_2) \left[ \int_{t_0}^{s_2} A(s_3) \, ds_3 \right] \, ds_2 \right] \, ds_1 + \ldots\]

The matrix function \(\Phi(t, t_0)\) is called the state transition function.

It satisfies the following properties

1. For every \(t\) and every \(t_0\), \(\Phi(t, t_0)\) is the unique solution of \(\frac{d}{dt} \Phi(t, t_0) = A(t)\Phi(t, t_0), \quad \Phi(t_0, t_0) = I\) proof by substitution

2. For any \(s\), \(\Phi(t, t_0) = \Phi(t, s)\Phi(s, t_0)\) [the semigroup property]

3. For every \(t\) and \(t_0\), \(\Phi(t, t_0)\) is nonsingular and since \(I = \Phi(t_0, t)\Phi(t, t_0) = \Phi(t, t_0)\Phi(t_0, t)\)
Solution to Nonhomogeneous LTV Systems

Theorem (*Variation of constants*)

The unique solution to

\[ \dot{x}(t) = A(t)x(t) + B(t)u(t), \quad y(t) = C(t)x(t) + D(t)u(t), \quad x(t_0) = x_0 \]

is

\[ x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau) \, d\tau \]

\[ y(t) = C(t)\Phi(t, t_0)x_0 + C(t) \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau) \, d\tau + D(t)u(t) \]

Proof by substitution again

Uniqueness follows by the global Lipschitz property of the mapping which is due to linearity

The zero-input response is \( y_{\text{zero-input}}(t) = C(t)\Phi(t, t_0)x_0 \)

The zero-state response is

\[ y_{\text{zero-state}}(t) = C(t) \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau) \, d\tau + D(t)u(t) \]
Discrete-time LTV Systems

The unique solution of \( x(t + 1) = A(t)x(t), \ x(t_0) = x_0, \ t \geq t_0, \ t, t_0 \in \mathbb{Z} \) is given by \( x(t) = \Phi(t, t_0)x_0, \ t \geq t_0 \)

where the discrete transition function, \( \Phi(t, t_0) \), is given by

\[
\Phi(t, t_0) = \begin{cases} 
I, & t = t_0 \\
A(t - 1)A(t - 2) \ldots A(t_0 + 1)A(t_0), & t > t_0 
\end{cases}
\]

Properties of \( \Phi(t, t_0) \)

1. \( \Phi(t, t_0) \) is the unique solution of
\[
\Phi(t + 1, t_0) = A(t)\Phi(t, t_0), \ \Phi(t_0, t_0) = I
\]

2. For every \( t \geq s \geq t_0 \), \( \Phi(t, t_0) = \Phi(t, s)\Phi(s, t_0) \)

There is no requirement for \( \Phi(t, t_0) \) to be nonsingular

\( \Phi(t, t_0) \) will be singular if any \( A(s), \ t > s \geq t_0, \) is singular

The unique solution of \( x(t + 1) = A(t)x(t) + B(t)u(t), \ x(t_0) = x_0 \)
\[
y(t) = C(t)x(t) + D(t)u(t)
\]
is
\[
x(t) = \Phi(t, t_0)x_0 + \sum_{\tau = t_0}^{t-1} \Phi(t, \tau + 1)B(\tau)u(\tau), \quad t \geq t_0
\]
\[
y(t) = C(t)\Phi(t, t_0)x_0 + \sum_{\tau = t_0}^{t-1} C(t)\Phi(t, \tau + 1)B(\tau)u(\tau) + D(t)u(t), \quad t \geq t_0
\]
Solution of LTI State-space Systems

The Matrix Exponential

Define for $M \in \mathbb{R}^{n \times n}$ its exponential $e^M = \sum_{k=0}^{\infty} \frac{1}{k!} M^k$ also in $\mathbb{R}^{n \times n}$

From the Peano-Baker series for the ODE $x(t_0) = x_0 \in \mathbb{R}^n$

$$\Phi(t, t_0) = I + \int_{t_0}^{t} A(s_1) \, ds_1 + \int_{t_0}^{t} \left[ \int_{t_0}^{s_1} A(s_2) \, ds_2 \right] \, ds_1$$

$$\quad + \int_{t_0}^{t} A(s_1) \left[ \int_{t_0}^{s_1} A(s_2) \left[ \int_{t_0}^{s_2} A(s_3) \, ds_3 \right] \, ds_2 \right] \, ds_1 + \ldots$$

we see that the transition function for the LTI system $(I-M)^{-1} = I + M + M^2 + M^3 + \ldots$ is

$$\Phi(t, t_0) = I + A \int_{t_0}^{t} \, ds_1 + A^2 \int_{t_0}^{t} \left[ \int_{t_0}^{s_1} \, ds_2 \right] \, ds_1 + A^3 \int_{t_0}^{t} \left[ \int_{t_0}^{s_1} \left[ \int_{t_0}^{s_2} \, ds_3 \right] \, ds_2 \right] \, ds_1 + \ldots$$

$$= I + A(t - t_0) + \frac{1}{2!} A^2(t - t_0)^2 + \frac{1}{3!} A^3(t - t_0)^3 + \ldots = e^{A(t-t_0)}$$

The solution to

$$x(t) = e^{A(t-t_0)} x_0 + \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau) \, d\tau$$

$$y(t) = C e^{A(t-t_0)} x_0 + \int_{t_0}^{t} C e^{A(t-\tau)} B u(\tau) \, d\tau$$
Properties of the Matrix Exponential

1. The function $e^{At}$ is the unique solution of \( \frac{d}{dt}e^{At} = Ae^{At}, \quad e^{A0} = I \)

2. For every $t, \tau \in \mathbb{R}$ we have $e^{At}e^{A\tau} = e^{A(t+\tau)}$
   but in general $e^{At}e^{Bt} \neq e^{(A+B)t}$ (unless $A$ and $B$ commute)

3. For every $t \in \mathbb{R}$, $e^{At}$ is nonsingular and $(e^{At})^{-1} = e^{-At}$

4. Using the Cayley-Hamilton theorem, for every $A \in \mathbb{R}^{n \times n}$, there exist $n$ scalar functions $\{\alpha_i(t) : i = 0, \ldots, n - 1\}$ such that
   \[
   e^{At} = \sum_{i=0}^{n-1} \alpha_i(t)A^i, \forall t \in \mathbb{R}
   \]

5. For every $A \in \mathbb{R}^{n \times n}$, $Ae^{At} = e^{At}A$

6. $e^{At} = \mathcal{L}^{-1} \left\{ (sI - A)^{-1}\right\} = \mathcal{L}^{-1} \left\{ \frac{1}{\det(sI - A)}[\text{adj}(sI - A)]^T \right\}$
   \[
   = \mathcal{L}^{-1} \left\{ \frac{1}{(s - \lambda_1)^{m_1} (s - \lambda_2)^{m_2} \ldots (s - \lambda_k)^{m_k}} \left[ \text{matrix of polynomials in } s \right] \right\}
   \]
   \[
   = [R_{11}]e^{\lambda_1 t} + [R_{12}]te^{\lambda_1 t} + \cdots + [R_{1m_1}]t^{m_1-1}e^{\lambda_1 t}
   + [R_{21}]e^{\lambda_2 t} + [R_{22}]te^{\lambda_2 t} + \cdots + [R_{2m_2}]t^{m_2-1}e^{\lambda_2 t}
   + \cdots + [R_{k1}]e^{\lambda_k t} + [R_{k2}]te^{\lambda_k t} + \cdots + [R_{km_k}]t^{m_k-1}e^{\lambda_k t}
   \]
   so $e^{At} \to 0$ as $t \to \infty$ if $\Re e[\lambda_i(A)] < 0, \quad i = 1, \ldots, n$
More on Matrix Exponentials

7. Since \((TAT^{-1})^k = TAT^{-1} \times TAT^{-1} \times \cdots \times TAT^{-1} = TA^k T^{-1}\)

\[e^{TAT^{-1}t} = \sum_{k=0}^{\infty} \frac{1}{k!} (TAT^{-1})^k t^k = T \left( \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k \right) T^{-1} = Te^{At} T^{-1}\]

for any invertible \(T\)

8. Using a generalized eigenvector matrix \(X\), \(X^{-1}AX = J\) (Jordan form)

\[e^{At} = X e^{Jt} X^{-1}\]

since \(J\) is block diagonal and upper triangular, so is \(e^{Jt}\)

This yields the same eigenvalue relationship as earlier

9. Discrete-time equivalent

Using the formal power series \((I - M)^{-1} = I + M + M^2 + M^3 + \ldots\)

\[Z^{-1} \left\{ (zI - A)^{-1} \right\} = Z^{-1} \left\{ z^{-1} (I - z^{-1} A)^{-1} \right\} = Z^{-1} \left\{ z^{-1} I + z^{-2} A + z^{-3} A^2 + \ldots \right\} \]

\[= Z^{-1} \left\{ \frac{1}{\det(zI - A)} [\text{adj}(zI - A)]^T \right\} \]

we have \(A^k \rightarrow 0\) as \(t \rightarrow \infty\) if \(|\lambda_i(A)| < 1, i = 1, \ldots n\)
Lyapunov Stability

**Definition** An equilibrium, $x^*$, of a system is stable in the sense of Lyapunov if given any $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that for all $x_0$ satisfying $|x^* - x_0| < \delta$ we have $|x^* - x(t)| < \epsilon$ for all $t > 0$

This is a definition applicable to all systems.

For linear systems it simplifies greatly:

$x^*=0$ is the candidate equilibrium.

Lyapunov stability is a property of the state equation only.

**Definition (Lyapunov stability for linear systems)**

\[
\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad y(t) = C(t)x(t) + D(t)u(t)
\]

(i) is (marginally) stable in the sense of Lyapunov if for every initial condition $x(t_0) = x_0 \in \mathbb{R}^n$ the homogeneous state response $x(t) = \Phi(t, t_0)x_0$, $t \geq t_0$ is uniformly bounded.

(ii) is asymptotically stable in the sense of Lyapunov if, in addition, for every initial condition $x(t_0) = x_0 \in \mathbb{R}^n$ we have $x(t) \to 0$ as $t \to \infty$.

(iii) is exponentially stable if, in addition, there exist constant $c, \lambda > 0$ such that for every initial condition $x(t_0) = x_0 \in \mathbb{R}^n$ we have

\[
\|x(t)\| \leq ce^{-\lambda(t-t_0)}\|x_0\| \quad \text{for all } t \geq t_0
\]

(iv) is unstable if it is not marginally stable in the sense of Lyapunov.

Lyapunov stability only treats the homogeneous system $\dot{x}(t) = A(t)x(t)$.
Lyapunov (internal) stability

Theorem (8.2)

The H-CLTI (homogeneous continuous-time linear time-invariant system)

\[ \dot{x}(t) = Ax(t), \quad x \in \mathbb{R}^n \]

(i) is marginally stable if and only if all the eigenvalues of \( A \) have negative or zero real parts and all Jordan blocks corresponding to eigenvalues with zero real parts are 1x1.

(ii) is asymptotically stable if and only if all eigenvalues of \( A \) have strictly negative real parts.

(iii) is exponentially stable if and only if all eigenvalues of \( A \) have strictly negative real parts.

(iv) is unstable if and only if at least one eigenvalue of \( A \) has positive real part or has zero real part but the corresponding Jordan block is larger than 1x1.

Proof

Follows from the properties of \( e^{At} \).

For LTI systems, asymptotic and exponential stability coincide.

This is NOT so for LTV systems - see any discussion of the Mathieu Equation.

\[ \ddot{y}(t) + [\alpha + \beta \cos t]y(t) = 0 \]
Stability of the Mathieu equation

The Mathieu equation - an LTV system

\[ \ddot{y}(t) + [\alpha + \beta \cos t]y(t) = 0 \]
Lyapunov Stability in General (Khalil)

\[ \dot{x}(t) = f(x) \]

Nonlinear, time-varying, homogeneous, autonomous, continuous-time system where \( f : D \to \mathbb{R}^n \) is locally Lipschitz from domain \( D \subset \mathbb{R}^n \) into \( \mathbb{R}^n \)

The equilibrium \( x^* \in D \), where \( f(x^*) = 0 \), is

(i) **stable** if for every \( \epsilon > 0 \) there is a \( \delta(\epsilon) > 0 \) such that
\[
\|x(0) - x^*\| < \delta \implies \|x(t) - x^*\| < \epsilon, \ \forall t \geq 0
\]

(ii) **unstable** if it is not stable

(iii) **asymptotically stable** if it is stable and \( \delta \) can be chosen so that
\[
\|x(0) - x^*\| < \delta \implies \lim_{t \to \infty} \|x(t) - x^*\| = 0
\]

(iv) **globally asymptotically stable** if it is asymptotically stable and \( \delta \) may be chosen arbitrarily

For nonautonomous systems \( \dot{x}(t) = f(x, t) \) we have similar ideas, but we must introduce the ideas of **uniform stability**

Note that for linear systems any kind of stability is necessarily a global property
Lyapunov continued

Lyapunov’s theorem

Let \( x=0 \) be an equilibrium of \( \dot{x}(t) = f(x) \) in the domain \( D \)

Let \( V : D \to \mathbb{R} \) be a continuously differentiable function such that

\[
V(0) = 0 \quad \text{and} \quad V(x) > 0 \quad \text{in} \quad D - \{0\} \quad \text{(positive definite)}
\]

\[
\dot{V}(x) \leq 0 \quad \text{in} \quad D \quad \text{(decrescent)}
\]

then \( x=0 \) is stable

Moreover, if \( \dot{V}(x) < 0 \) in \( D - \{0\} \) then \( x=0 \) is asymptotically stable

If \( D = \mathbb{R}^n \) and \( ||x|| \to \infty \implies V(x) \to \infty \) (radially unbounded) then \( x=0 \) is globally asymptotically stable

\( V(x) \) is a scalar function of the \( n \)-vector \( x \)

It represents something akin to energy for a dissipative system

Sometimes it is easier to find \( V \) than it is to prove stability directly

There are many other uses for Lyapunov functions e.g. domain of attraction estimation
Lyapunov Stability in Linear Systems

Consider the continuous-time LTI system \( \dot{x} = Ax \)

The following statements are equivalent

(i) The system is asymptotically stable
(ii) The system is globally asymptotically stable
(iii) The system is globally exponentially stable
(iv) All the eigenvalues of \( A \) have negative real parts
(v) For every symmetric positive definite matrix \( Q \) there exists a unique solution to the (continuous-time) Lyapunov Equation

\[
A^T P + PA = -Q
\]

Moreover, \( P \) is symmetric and positive definite

(vi) There exists a symmetric positive definite matrix \( P \) for which the Lyapunov matrix inequality holds \( A^T P + PA < 0 \)

If \( A \) is asymptotically stable the unique solution of the Lyapunov Equation is

\[
P = \int_0^\infty e^{A^T t} Q e^{A t} \, dt
\]

\( V(x) = x^T P x \) is a Lyapunov function for this system
Discrete-time Linear Lyapunov Stability

The discrete-time LTV system \( x(t + 1) = A(t)x(t) \) is

(i) marginally stable in the sense of Lyapunov if for every initial condition \( x(t_0) = x_0 \) the homogeneous state response \( x(t) = \Phi(t, t_0)x_0 \) is uniformly (in \( t \) and \( t_0 \)) bounded

(ii) asymptotically stable in the sense of Lyapunov if, in addition, we have for every initial condition \( x(t) \to 0 \) as \( t \to \infty \)

(iii) exponentially stable if there exist constants \( c, \lambda \) such that for every initial condition \( \|x(t)\| \leq c\lambda^{(t-t_0)}\|x_0\|, \forall t \geq t_0 \)

(iv) unstable if it is not marginally stable in the sense of Lyapunov

The homogeneous discrete-time LTI system \( x(t + 1) = Ax(t) \) is

(i) marginally stable if and only if all the eigenvalues of \( A \) have magnitude less than or equal to one and all the Jordan blocks corresponding to eigenvalues equal to one have size 1x1

(ii) asymptotically and exponentially stable if and only if all of the eigenvalues of \( A \) have magnitude less than one

(iii) unstable if and only if at least one eigenvalue of \( A \) has magnitude larger than one or magnitude equal to one and Jordan block larger than 1x1
Discrete-time Linear Stability

For the discrete-time homogeneous LTI system $x(t + 1) = Ax(t)$ the following statements are equivalent

(i) The system is asymptotically stable
(ii) The system is exponentially stable
(iii) All eigenvalues of $A$ have magnitude less than one
(iv) For every symmetric positive definite matrix $Q$ there exists a unique solution $P$ to the Stein Equation (Discrete-time Lyapunov Equation)

$$A^T PA - P = -Q$$

Moreover, $P$ is symmetric and positive definite

(v) There exists a symmetric positive definite matrix $P$ for which the following Lyapunov matrix inequality holds $A^T PA - P < 0$

For stable $A$, use the solution

$$P = \sum_{j=0}^{\infty} A^j Q A^j$$

$V(x) = x^T Px$ is a discrete-time Lyapunov function for this system.
Discrete-time Lyapunov Stability

Consider the discrete-time nonlinear homogeneous autonomous system

\[ x(t + 1) = f(x(t)) \]

Let \( x=0 \) be an equilibrium in \( D \), as before

Let \( V : D \to \mathbb{R} \) be a function such that along all solutions of the system

\[
V(0) = 0 \text{ and } V(x) > 0 \text{ in } D - \{0\} \quad \text{(positive definite)}
\]

\[
V(x(t + 1)) = V(f(x(t))) \leq V(x(t)) \text{ in } D \quad \text{(decrescent)}
\]

then \( x=0 \) is stable

Moreover, if \( \Delta V(x(t)) = V(f(x(t))) - V(x(t)) < 0 \) in \( D - \{0\} \) then \( x=0 \) is asymptotically stable

If \( D = \mathbb{R}^n \) and \( V(x) \) is radially unbounded then \( x=0 \) is g.a.s.

Very similar ideas to continuous time

Uses the Monotone Convergence Theorem
Consider the discrete-time LTI system \( x(t + 1) = Ax(t) \)

Take as our candidate Lyapunov function \( V(x) = x^T P x \) with \( P > 0 \) satisfies positive definite and radially unbounded properties

Suppose \( A^T P A - P = -C^T C \) where \( C \) is \( m \times n \) so that the right-hand side of the discrete Lyapunov equation is only nonpositive definite

Then \( V[x(t + 1)] - V[x(t)] = x(t + 1)^T P x(t + 1) - x(t)^T P x(t) \)

\[
= x(t)^T \left[ A^T P A - P \right] x(t)
= -x(t)^T C^T C x(t) = [C x(t)]^T [C x(t)] \leq 0
\]

But

\[
V[x(t + n)] - V[x(t)] = V[x(t + n)] - V[x(t + n - 1)] + V[x(t + n - 1)] - V[(x(t + n - 2)]
+ \cdots + V[x(t + 1)] - V[x(t)]
\]

\[
= -x(t + n - 1)^T C^T C x(t + n - 1) - x(t + n - 2)^T C^T C x(t + n - 2)
- \cdots - x(t)^T C^T C x(t)
\]

\[
= -x(t)^T \left[ A^{(n-1)T} C^T C A^{(n-1)} + A^{(n-2)T} C^T C A^{(n-2)} + \cdots + C^T C \right] x(t)
= -x(t)^T [O^T O] x(t)
< 0, \quad \text{if } [C, A] \text{ is observable}
\]
A Grown-ups’ Version of the Lyapunov Equation

The continuous-time LTI system \( \dot{x} = Ax \) is g.a.s. if there exists a positive definite solution \( P \) to the Lyapunov matrix equation

\[
A^T P + PA = -C^T C
\]

with the matrix pair \([A, C]\) observable

The discrete-time LTI system \( x(t + 1) = Ax(t) \) is g.a.s. if there exists a positive definite solution to the discrete-time Lyapunov matrix equation

\[
A^T PA - P = -C^T C
\]

with the matrix pair \([A, C]\) observable

We see observability arise again

It has to do with the only solutions to the equation which yield zero derivative (difference) over a time interval are those which are zero

\( V \) is positive definite and decreasing, so it converges

So \( \Delta V \) converges to zero and this can only happen if \( x \rightarrow 0 \)
**Input-Output Stability**

\[ \dot{x}(t) = A(t)x(t) + B(t)u(t), \quad y(t) = C(t)x(t) + D(t)u(t), \quad x(t_0) = x_0 \]

The response of this system is \( y(t) = y_{\text{zero-input}}(t) + y_{\text{zero-state}}(t) \) where

\[
y_{\text{zero-input}}(t) = C(t)\Phi(t, t_0)x_0
\]

\[
y_{\text{zero-state}}(t) = \int_{t_0}^{t} C(t)\Phi(t, \tau)B(\tau)\,d\tau + D(t)u(t)
\]

Lyapunov deals with the first term provided \( C(t) \) is uniformly bounded

Bounded-Input Bounded-Output (BIBO) stability deals with the second term

**Definition:** The continuous-time LTV system is *uniformly BIBO stable* if there exists a constant \( g \) such that for every input \( u(t) \) its zero-state response satisfies

\[
\text{sup}_{t \in [t_0, \infty)} \| y_{\text{zero-state}}(t) \| \leq g \text{ sup}_{t \in [t_0, \infty)} \| u(t) \|
\]

Note that these are just (euclidean) vector norms not function norms

The LTV system is BIBO stable if and only if every element of \( D(t) \) is uniformly bounded and

\[
\text{sup}_{t \in [t_0, \infty)} \int_{t_0}^{t} \left| [C(t)\Phi(t, \tau)B(\tau)]_{ij} \right|\,d\tau < \infty
\]
Continuous-time LTI BIBO Stability

The system \( \dot{x} = Ax + Bu, \ y = Cx + Du \) is BIBO stable if and only if
\[
\int_0^\infty \left| [Ce^{Ap} B]_{ij} \right| \, dp < \infty
\]

This condition is met if and only if every element of the transfer function matrix \( G(s) = D + C(sI - A)^{-1}B \) only has poles with negative real part

When the system is exponentially stable then it must also be BIBO stable

The reason for the rather peculiar condition on the transfer function is that we need to be concerned with unstable pole-zero cancellations

We saw some examples of systems with \( A \) matrices which were unstable while the transfer function was stable

\[
A = \begin{bmatrix} -4 & -1 & 6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}
\]

\( \lambda_i(A) = \{-3, -2, 1\} \), Lyapunov unstable

\[
G(s) = \frac{s^2 - 1}{s^3 + 4s^2 + s - 6} = \frac{(s + 1)(s - 1)}{(s + 2)(s + 3)(s - 1)} = \frac{s + 1}{(s + 2)(s + 3)} \quad \text{BIBO stable}
\]
Discrete-time Input-output Stability

\[ x(t + 1) = A(t)x(t) + B(t)u(t), \quad y(t) = C(t)x(t) + D(t)u(t) \]

Zero-state response

\[ y_{zs}(t) = \sum_{\tau=0}^{t-1} C(t)\Phi(t, \tau + 1)B(\tau)u(\tau) + D(t)u(t), \quad \forall t \geq 0 \]

The discrete-time LTV system is uniformly \textit{BIBO stable} if there exists a constant \( g \) such that \( \sup_{t \in \mathbb{N}} \|y_{zs}(t)\| \leq g \sup_{t \in \mathbb{N}} \|u(t)\| \)

The system is uniformly BIBO stable if and only if

\[ \sup_{t \in \mathbb{N}} \sum_{\tau=0}^{t-1} \left| [C(t)\Phi(t, \tau)B(\tau)]_{ij} \right| < \infty, \quad \text{for } i = 1, \ldots, m, \ j = 1, \ldots, k \]

For the discrete-time LTI system \( x^+ = Ax + Bu, \ y = Cx + Du \)

the following statements are equivalent

(i) the system is uniformly BIBO stable

(ii) \[ \sum_{\tau=0}^{\infty} \left| [CA^\tau B]_{ij} \right| < \infty, \quad \text{for } i = 1, \ldots, m, \ j = 1, \ldots, k \]

(iii) all the poles of all the elements of the transfer function matrix lie strictly inside the unit circle.
Summary

The ideas of LTV systems specialize in the case of LTI systems

Likewise, many ideas from linear systems give guidance how to proceed with nonlinear systems

Lyapunov stability is one example

The concepts of an energy-like function and of dissipation are entirely generalizable
Finding a Lyapunov function for a nonlinear system is often much simpler than proving stability by any other means
The Lyapunov function can help find the region of attraction of an equilibrium

Input-output stability differs dramatically from Lyapunov stability
Ideas from normed function spaces come to the fore
MAE281A, 281B begin to get a handle on all of this

That is why we spent more time on these ideas